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A THRESHOLD THEORY FOR PHASE-LOCKED LOOPS

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Group 25

TECHNICAL REPORT NO. 246

22 AUGUST 1961

ABSTRACT

A model of a phase-locked loop has been developed which is valid for all signal-to-noise ratios. The model is in the form of a nonlinear feedback system with randomly time-varying parameters. The analysis considers two operating regions.

In low signal-to-noise ratio regions, the important consideration is stability. We want to study the asymptotic stability in the mean of a nonlinear system. It follows directly that a necessary condition for asymptotic stability of any nonlinear system is that a linearized model about some equilibrium point be asymptotically stable. By considering all possible equilibrium points, we can find an upper bound on the value of noise density which makes the system unstable. This upper bound represents a threshold value for system operation.

In high signal-to-noise ratio regions, our results provide an exact statistical description of system behavior. Therefore, knowledge of the spectrum of the signal and noise may be used to optimize the system configuration.

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A THRESHOLD THEORY FOR PHASE-LOCKED LOOPS

I. INTRODUCTION

The majority of modern communication systems employ coherent detection. Implementing this technique requires that the phase of the carrier be known. At present, the most commonly used system for phase detection is a phase-locked loop.

A typical loop is shown in Fig. 1. The input to the system is a sine wave whose phase is the quantity of interest plus additive noise. The input is multiplied by a feedback signal which is the output of a voltage-controlled oscillator (VCO). This output is a cosine wave of the same frequency whose phase θ_2 is the estimate of θ_1.

The output of the multiplier is

$$E(t) = A \sin(\theta_1 - \theta_2) + \sqrt{2}\, N'(t) \cos(\omega t + \theta_2) \quad , \tag{1}$$

where the double frequency terms have been deleted, since the filter and VCO would not respond to them.

This error term is passed through a filter. The output of the filter is a voltage which controls the instantaneous frequency of the oscillator. The phase of the VCO provides the necessary reference.

The operation and optimization of phase-locked loops have been extensively discussed in the literature. The results of the majority of these analyses[1,2] are linearized models which are valid only for high signal-to-noise ratios and low values of E(t). Margolis[3] uses a series expansion technique, but the convergence of the series is directly related to the signal-to-noise ratio. More recently, Viterbi[4] has studied the nonlinear dynamic behavior for noise-free systems.

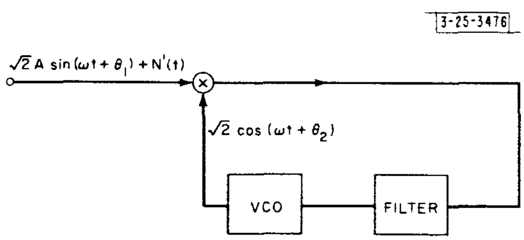


Fig. 1. Phase-locked loop.

As the noise level increases, the error increases, until finally the system no longer follows the input. Qualitatively, the concept of a threshold below which the system operates asynchronously is quite logical. Many authors have attempted to analyze this threshold quantitatively by direct extension of the simple linear model. Unfortunately, these linear models have already eliminated what appears to be the actual cause of thresholding.

The results of the research outlined in this report are twofold. First, a complete model for a phase-locked loop with a noisy input is derived. This model is a nonlinear feedback system with randomly time-varying parameters.

Two regions of operation are then considered. When the noise-to-signal ratio is high, the primary concern is stability. In Secs. III and IV, we develop bounds on the noise-to-signal ratio for stable operation. When the noise-to-signal ratio is low, the system can be optimized to achieve minimum mean-square phase error. The optimization technique is outlined in Sec. V.

II. MODEL OF A PHASE-LOCKED LOOP

Let the noise $N'(t)$ be a sample function from a white, narrow-band Gaussian process. One may then write[5] (see Appendix A)

$$N'(t) = m_1'(t) \sin \omega_1(t) + m_2'(t) \cos \omega_1 t \quad , \tag{2}$$

where $m_1'(t)$ and $m_2'(t)$ are sample functions from independent, band-limited, white noise processes.

$$\overline{m_1'(t)} = \overline{m_2'(t)} = 0 \quad , \tag{3}$$

and

$$\overline{m_1(t)^2} = \overline{m_2(t)^2} = 2N_o' \, W \quad , \tag{4}$$

where N_o' and W are defined in Appendix A.

Equally well, if θ_1 is fixed one could write

$$N'(t) = n_1'(t) \sin(\omega_1 t + \theta_1) + n_2'(t) \cos(\omega_1 t + \theta_1) \quad . \tag{5}$$

Clearly, $n_1'(t)$ and $n_2'(t)$ have exactly the same properties as $m_1'(t)$ and $m_2'(t)$.

The two pairs of random variables are related by a rotational transformation.

$$\begin{aligned} m_1'(t) &= n_1'(t) \cos \theta_1 - n_2'(t) \sin \theta_1 \quad , \\ m_2'(t) &= n_1'(t) \sin \theta_1 + n_2'(t) \cos \theta_1 \quad . \end{aligned} \tag{6}$$

The decompositions in Eqs. (2) and (5) suggest the two models of phase-locked loops shown in Figs. 2 and 3. In each case, the actual loop is shown first and then redrawn with phase as the variable. One sees that the additive Gaussian noise at the input manifests itself as a random gain variation in the phase-locked loop.

We now have two models which provide an exact description of system behavior. Let us first consider the problem of system stability.

In a nonlinear system, we must define precisely what we mean by stability. Consider first a general deterministic system with an equilibrium solution described by some state vector $\underline{x}_\epsilon(t)$. Normally, the components of this state vector $\underline{x}_\epsilon(t)$ will be the position x_ϵ, the velocity $\dot{x}_\epsilon$, and the various derivatives up to $x_\epsilon^{(n)}$. Similarly, the instantaneous state is described by a vector $\underline{x}(t)$. A convenient norm or distance measure is just the Euclidean distance between the two vectors. Thus, we define

$$||\underline{x}(t_1) - \underline{x}_\epsilon(t_1)|| = [\underline{x}(t_1) - \underline{x}_\epsilon(t_1)]^2 \quad .$$

Then, one says the system is stable, if for any $\epsilon \geq 0$ there exists a $\delta(\epsilon) > 0$, such that

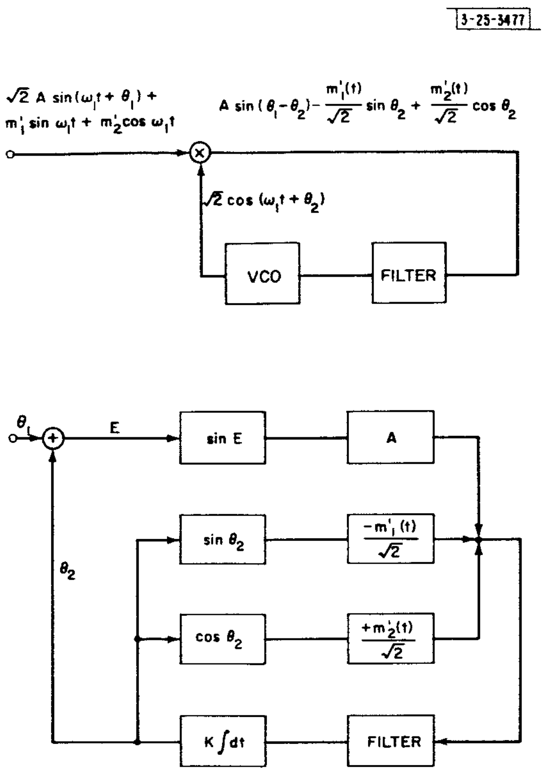


Fig. 2. Model I.

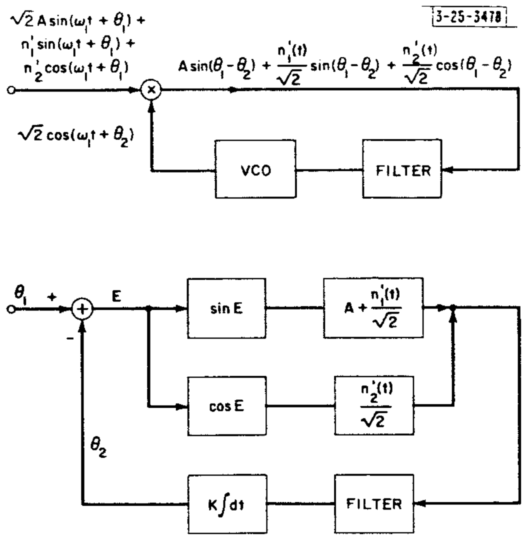


Fig. 3. Model II.

$||\underline{x}(0) - \underline{x}_\epsilon(0)|| < \delta(\epsilon)$ implies $||\underline{x}(t) - \underline{x}_\epsilon(t)|| < \epsilon$ for all $t \geq 0$. A system is asymptotically stable if there also exists a $\delta_1(0) > 0$ such that

$$\lim_{t \to \infty} ||\underline{x}(t) - \underline{x}_\epsilon(t)|| = 0 \quad .$$

These definitions of stability correspond to our intuitive notions. We perturb the system from its equilibrium state. If it is stable, it will return to this equilibrium state.

The concept of stability in randomly time-varying systems follows directly.[7] We can say a system is stable in the mean if $\langle ||x(t) - x_\epsilon(t)|| \rangle$ satisfies the above conditions.

Clearly, a necessary, but not sufficient, condition for stability in the mean about some $\underline{x}_\epsilon(t)$ is that the linearized model about $\underline{x}_\epsilon(t)$ be stable. Our approach to finding a bound on the noise threshold is to consider the linearized model about all possible equilibrium points. We will show that about a certain value of N_o

$$\lim_{t \to \infty} \langle ||x(t) - x_\epsilon(t)|| \rangle = \infty \quad .$$

This value of N_o then specifies a threshold above which the system operates asynchronously.

A logical first approach is to investigate a linearized, randomly time-varying model in the region where $E(t)$ is small.

III. FIRST-ORDER SYSTEMS

Consider first a simple system whose input is a fixed-frequency sinusoid whose frequency differs from that of the free-running voltage-controlled oscillator. Using the model of

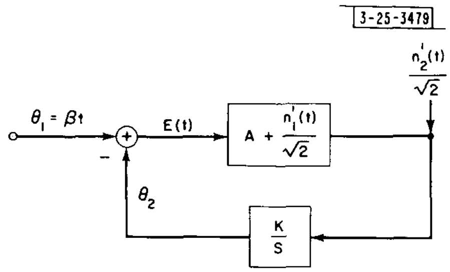


Fig. 4. Linearized first-order loop.

Fig. 3 and linearizing about $E(t) = 0$, we have the system of Fig. 4.

The assumptions are:

(a) $\sin(\theta_1 - \theta_2) \approx \theta_1 - \theta_2$,

(b) $\cos(\theta_1 - \theta_2) \approx 1$,

(c) Filter has unity transfer function,

(d) $\theta_1 = \beta t$, where β is defined as the difference between the input frequency and the free-running VCO frequency.

The differential equations describing the system are:

$$E(t) = \theta_1 - \theta_2 = \beta t - \theta_2 , \tag{7}$$

$$\theta_2 = AK' \, E(t) \left[1 + \frac{n_1'(t)}{\sqrt{2} \, A} \right] + K' \frac{n_1'(t)}{\sqrt{2}} . \tag{8}$$

Combining (7) and (8), and defining

$$n_1(t) = \frac{n_1'(t)}{\sqrt{2} \, A} , \quad n_2(t) = \frac{n_2'(t)}{\sqrt{2} \, A} , \quad \text{and } K = K' A ,$$

one has

$$\beta \mu_{-1}(t) - \dot{E}(t) = K \, E(t) \left[1 + n_1(t) \right] + K n_2(t) . \tag{9}$$

Rearranging,

$$E(t) + E(t) \left[K \left[1 + n_1(t) \right] \right] = \beta \mu_{-1}(t) + K n_2(t) . \tag{10}$$

Using the integrating factor technique, one has

$$E(t) = \exp \left\{ -K \int_0^t 1 + n_1(\tau) \, d\tau \right\} \cdot \int_0^t \left[\beta \mu_{-1}(x) + K n_2(x) \right] \exp \left\{ +K \int_0^x \left[1 + n_1(\tau) \right] d\tau \right\} dx . \tag{11}$$

Rewriting to separate the deterministic and random terms,

$$E(t) = \int_0^t [\beta + Kn_2(x)] \exp[K(x-t)] \exp\left[-K\int_x^t n_1(\tau)\, d\tau\right] dx \, . \tag{12}$$

To characterize system behavior, we desire to find the mean and variance of E(t).

$$\langle E(t) \rangle = \left\langle \int_0^t [\beta + Kn_2(x)] \exp[K(x-t)] \exp\left[-K\int_x^t n_1(\tau)\, d\tau\right] dx \right\rangle \, . \tag{13}$$

Noting that $n_2(t)$ and $n_1(t)$ are independent processes and $\langle n_2(t) \rangle = 0$, Eq. (13) becomes

$$\langle E(t) \rangle = \beta \int_0^t \exp[K(x-t)] \left\langle \exp\left[-K\int_x^t n_1(\tau)\, d\tau\right] \right\rangle dx \, . \tag{14}$$

In Appendix B, it is shown that

$$\left\langle \exp\left[-K\int_0^t n_1(\tau)\, d\tau\right] \right\rangle = \exp\left[\frac{K^2 N_o (t-x)}{2}\right] \tag{15}$$

for $n_1(\tau)$ being a band-limited white Gaussian process whose double-sided spectral density has a height N_o. Therefore,

$$\langle E(t) \rangle = \beta \int_0^t \exp[K(x-t)] \exp\left[\frac{K^2 N_o [t-x]}{2}\right] dx \, , \tag{16}$$

$$\langle E(t) \rangle = \frac{\beta}{K} \cdot \frac{1}{1 - \frac{KN_o}{2}} \left\{ 1 - \exp\left[-K\left(1 - \frac{KN_o}{2}\right) t \right]\right\} \tag{17}$$

One notes that for $KN_o > 2$, $\langle E(t) \rangle$ is a monotone increasing function of time.

$$\begin{aligned}\langle E^2(t) \rangle = \Bigg\langle & \int_0^t [\beta\mu_{-1}(x) + Kn_2(x)] \exp[K(x-t)] \exp\left[-K\int_x^t n_1(\tau)\, d\tau\right] dx \\ & \cdot \int_0^t [\beta\mu_{-1}(y) + Kn_2(y)] \exp[K(y-t)] \exp\left[-K\int_y^t n_1(\tau)\, d\tau\right] dy \Bigg\rangle \, ,\end{aligned} \tag{18}$$

$$\begin{aligned}\langle E^2(t) \rangle = {} & \beta^2 \exp[-2Kt] \int_0^t dx \int_0^t dy \cdot \exp[K(x+y)] \cdot \left\langle \exp\left[-K\int_x^t n_1(\tau) d\tau\right] \cdot \exp\left[-K\int_y^t n_1(\tau) d\tau\right]\right\rangle \\ & + K^2 \exp[-2Kt] \int_0^t dx \left\langle \int_0^t dy \cdot N_o \, \delta(x-y) \exp[K(x+y)] \right. \\ & \left. \cdot \exp\left[-K\int_x^t n_1(\tau)\, d\tau - K\int_y^t n_1(\tau)\, d\tau\right] \right\rangle \, .\end{aligned} \tag{19}$$

Rearranging the first term so that the integration intervals are disjoint, we have

$$\left\langle E^2(t)\right\rangle = 2\beta^2 \exp[-2Kt] \int_0^t dx \int_0^x dy \cdot \exp[K(x+y)] \cdot \left\langle \exp\left[-2K \int_x^t n_1(\tau)d\tau - K \int_y^x n_1(\tau)d\tau\right]\right\rangle$$

$$+ K^2 N_o \exp[-2Kt] \int_0^t dx \cdot \exp[+2Kx] \cdot \left\langle \exp\left[-2K \int_x^t n_1(\tau)\, d\tau\right]\right\rangle . \tag{20}$$

Now the random variables in the first term are independent and therefore their joint expectation factors. This assumes W is large enough that $n_1(\tau)$ is essentially white.

$$\left\langle E^2(t)\right\rangle = 2\beta^2 \exp[-2Kt] \int_0^t dx \int_0^t dy \cdot \exp[K(x+y)] \exp[+2K^2N_o(t-x)] \cdot \exp\left[\frac{+K^2 N_o (x-y)}{2}\right]$$

$$+ K^2 N_o \exp[-2Kt] \exp[+2K^2 N_o t] \int_0^t dx \exp[[+2K - 2K^2 N_o]\, x] . \tag{21}$$

Integrating, one has

$$\left\langle E^2(t)\right\rangle = 2\frac{\beta^2}{K^2} \cdot \frac{1}{\left(1 - \frac{KN_o}{2}\right)} \left\{ \frac{1 - \exp[-2K(1 - KN_o)t]}{2(1 - KN_o)} - \frac{\exp\left[-K\left(1 - \frac{KN_o}{2}\right) t\right] - \exp[-2K(1-KN_o)t]}{1 - \frac{3}{2}KN_o} \right\}$$

$$+ \frac{KN_o}{2(1 - KN_o)} \{1 - \exp[-2K(1 - KN_o)t]\} . \tag{22}$$

For $KN_o > 1$, the variance becomes unbounded for large t.

For $KN_o < 1$, as $t \to \infty$, one has

$$\left\langle E(t)\right\rangle = \frac{\beta}{K} \cdot \frac{1}{1 - \frac{KN_o}{2}} , \tag{23}$$

and

$$\left\langle E^2(t)\right\rangle = \frac{\beta^2}{K^2} \cdot \frac{1}{\left(1 - \frac{KN_o}{2}\right)(1 - KN_o)} + \frac{KN_o}{2(1 - KN_o)} . \tag{24}$$

Thus, investigation of the linear, random parameter model gives three results:

(a) A noise level, $N_o > 2/K$, at which the mean becomes unbounded for large t,

(b) A noise level, $N_o > 1/K$, at which the variance becomes unbounded for large t,

(c) An expression for the mean and variance which is valid for small E(t).

Upon reviewing the literature, we found that the idea of an unbounded variance in a random, linear system has been pointed out previously by Rosenbloom.[6] His approach was somewhat different, but the end results agree. He also considers the case where $n_1(\tau)$ is not white.

Clearly, if β/K or KN_o approach one, our linear model about the origin is inadequate to describe system behavior.

The next step is to investigate the behavior of the mean and variance as E(t) varies over

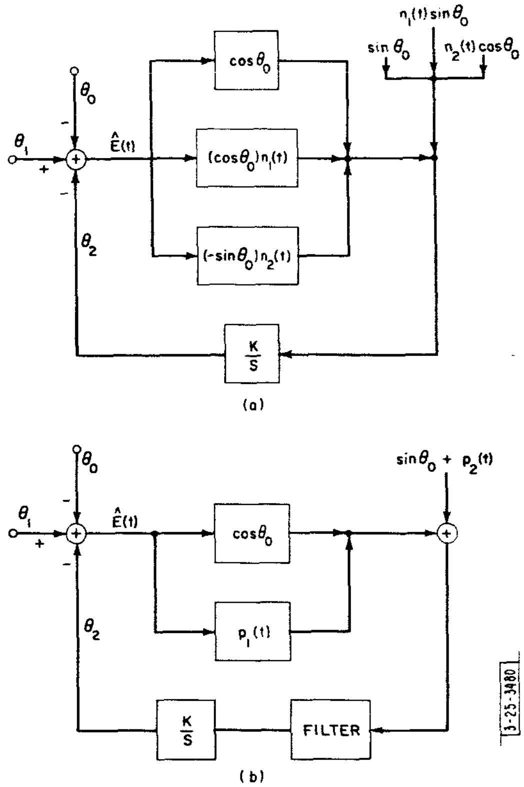


Fig. 5. Model linearized about an arbitrary equilibrium point.

all possible equilibrium points.

Our approach is to select some operating point θ_o. Then, by varying θ_o, we can study the behavior over the complete range E(t).

In the vicinity of θ_o,

$$\sin E = \sin\theta_o + \cos\theta_o\,(E - \theta_o) \quad , \tag{25}$$

$$\cos E = \cos\theta_o - \sin\theta_o\,(E - \theta_o) \quad . \tag{26}$$

Therefore, in the region around θ_o, the model of Fig. 3 can be redrawn as shown in Fig. 5(a).

We have defined

$$\widehat{E}(t) = E(t) - \theta_o \quad . \tag{27}$$

One notes that there are two random terms in our model: a random gain term, $P_1(t)$, where

$$P_1(t) = n_1(t)\cos\theta_o - n_2(t)\sin\theta_o \quad , \tag{28}$$

and a random input term, $P_2(t)$, where

$$P_2(t) = n_1(t)\sin\theta_o + n_2(t)\cos\theta_o \quad . \tag{29}$$

But one observes that Eqs. (28) and (29) are identical in form to Eq. (6). Thus, $P_1(t)$ and $P_2(t)$

are just sample functions from two independent random processes, and the model can be redrawn as in Fig. 5(b). The differential equation describing the system is

$$\beta\mu_{-1}(t) - \dot{\hat{E}}(t) = K\,E(t)\,[\cos\theta_o + P_1(t)] + K[\sin\theta_o + P_2(t)] \quad . \tag{30}$$

Proceeding as before, we have

$$\hat{E}(t) = \int_0^t [(\beta - K\sin\theta_o)\,\mu_{-1}(x) + KP_2(x)]\exp[K\cos\theta_o(x-t)]\,\exp\left[-K\int_x^t P_1(\tau)\,d\tau\right] \quad . \tag{31}$$

Identifying analogous quantities, we can write

$$\langle E(t)\rangle = \frac{\beta - K\sin\theta_o}{K}\cdot\frac{1}{\left(\cos\theta_o - \frac{KN_o}{2}\right)}\left\{1 - \exp\left[-K\left(\cos\theta_o - \frac{KN_o}{2}\right)t\right]\right\} , \tag{32}$$

and

$$\hat{E}(t)^2 = 2\frac{(\beta - K\sin\theta_o)^2}{K^2}\cdot\frac{1}{\left(\cos\theta_o - \frac{KN_o}{2}\right)}\left\{\frac{1 - \exp[-2K(\cos\theta_o - KN_o)t]}{2(\cos\theta_o - KN_o)}\right.$$

$$\left. - \frac{\exp\left[-K\left(\cos\theta_o - \frac{KN_o}{2}\right)t\right] - \exp[-2K(\cos\theta_o - KN_o)t]}{\left[\cos\theta_o - \frac{3}{2}KN_o\right]}\right\}$$

$$+ \frac{KN_o}{2(\cos\theta_o - KN_o)}\{1 - \exp[-2K(\cos\theta_o - KN_o)t]\} \quad . \tag{33}$$

Examining the expression for $\langle\hat{E}(t)\rangle$, we see that for $\beta = K\sin\theta_o$, the mean is stationary, regardless of the noise level. One can show easily that this stationary point is stable for perturbations.

Thus,

$$\lim_{t\to\infty}\langle E(t)\rangle = \lim_{t\to\infty}\langle\hat{E}(t)\rangle + \theta_o = \sin^{-1}\frac{\beta}{K} \quad . \tag{34}$$

One can make several observations at this point. First, in order to have a stable mean, the inequality

$$\beta < K \tag{35}$$

must be satisfied. This is exactly the limitation of the noiseless case.[4] Secondly, by refining our analysis, we have obtained an expression for the mean that is accurate for moderate KN_o. It is important to notice that, although the stationarity of the mean did not depend on KN_o, our linearized model does not give an exact quantitative description of the loop's nonlinear behavior for large KN_o.

We see that $\langle E^2(t)\rangle$ becomes unbounded for large t when $KN_o > \cos\theta_o$.

For $KN_o < \cos\theta_o$,

$$\lim \langle \hat{E}^2(t) \rangle = \frac{KN_o}{2(\cos \theta_o - KN_o)} \quad , \tag{36}$$

and

$$\sigma^2 [\hat{E}(t)] = \langle \hat{E}^2(t) \rangle \quad . \tag{37}$$

This accurately describes the nonlinear system variance for moderate KN_o.

It is worth while to stop and interpret our results for the first-order loop. We have shown that linearizing about any E(t), the inequality $KN_o > 1$ leads to an unbounded variance. Thus, this inequality may be interpreted as an absolute threshold for asynchronous operation. Clearly, even for $KN_o < 1$, the loop will slip cycles occasionally. However, when KN_o exceeds 1 the variance of the error will become unbounded.

In practice, most loops have a filter. Since one cannot obtain an explicit solution for a second or higher-order differential equation, our straightforward technique fails. Fortunately, Rosenbloom[6] has developed an expression for the mean and variance of the output of a randomly varying linear system under certain restrictions. This technique will enable us to study a certain class of higher-order systems.

IV. HIGHER-ORDER SYSTEMS

Consider the system in Fig. 5(b) when the filter is a linear system whose transfer function is

$$H(s) = \frac{1}{F(s)} \quad , \tag{38}$$

where F(s) is a polynomial.

For a fixed frequency offset, the equations describing the system are

$$\theta_1 - \theta_2 = \beta t - \theta_2 = E(t) = \theta_o + \hat{E}(t) \quad , \tag{39}$$

$$\frac{p\,F(p)}{K}\,\theta_2 = \hat{E}(t)\,[\cos \theta_2 + p_1(t)] + \sin \theta_o + p_2(t) \quad , \tag{40}$$

where $p = d/dt$.

Combining gives

$$\frac{p\,F(p)}{K} \cdot [\beta t - \theta_o - \hat{E}(t)] = \hat{E}(t)\,[\cos \theta_o + p_1(t)] + \sin \theta_o + p_2(t) \quad . \tag{41}$$

This equation may be rewritten as

$$L(p)\,E(t) + p_1(t)\,\hat{E}(t) = x(t) \quad , \tag{42}$$

where

$$L(p) = \frac{p\,F(p)}{K} + \cos \theta_o \quad ,$$

$$x(t) = \frac{p\,F(p)}{K} \cdot (\beta t - \theta_o) - \sin \theta_o - p_2(t) \quad .$$

Here L(p) is a deterministic differential operator, $p_1(t)$ is a sample function from a zero-mean, white Gaussian process, and x(t) is the sum of a random process and a deterministic component.

In Appendix C, the steps leading to an expression for the mean and variance are outlined. The pertinent results are shown below.

We define $\ell(t)$ as the impulse response corresponding to the differential operator L(P). Thus,

$$\ell(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \frac{1}{L(s)} \cdot e^{+st}\, ds = \mathcal{L}^{-1}\left[\frac{1}{L(s)}\right] . \tag{43}$$

We define

$$\Lambda(s) = \mathcal{L}[\ell^2(t)] \equiv \int_0^\infty \ell^2(t)\, e^{-st}\, dt \quad . \tag{44}$$

It is shown in Appendix C that

$$\left\langle \hat{E}(t) \right\rangle = \left\langle E_o(t) \right\rangle \quad , \tag{45}$$

and

$$\left\langle \hat{E}^2(t) \right\rangle = \mathcal{L}^{-1}\left[\frac{\mathcal{L}\left\langle E_o^2(t)\right\rangle}{1 - N_o\Lambda(s)}\right] \quad . \tag{46}$$

Clearly, if the function $1 - N_o \Lambda(s)$ has roots in the right half plane, the variance will be unstable.

If $\lim_{t\to\infty} \left\langle \hat{E}(t)\right\rangle$ and $\lim_{t\to\infty} \left\langle \hat{E}(t)^2\right\rangle$ exists, one can find the spectral density of $\hat{E}(t)$ for large t. From Appendix C, we have

$$\lim_{t\to\infty} S_{\hat{E}}(\omega) = \left[S_x(\omega) + \frac{\sigma^2_{\hat{E}_o}(\infty)\, \frac{N_o}{2\pi}}{1 - N_o\, \Lambda(O)} \right] |L(j\omega)|^2 \quad . \tag{47}$$

We notice that the first term is identical to that obtained by a stationary, linear analysis. The quantity $\Lambda(O)$ is readily interpretable by applying the final value theorem to Eq. (47).

$$\Lambda(O) \equiv \Lambda(s)\Big|_{s=0} = \int_0^\infty \ell^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |L(j\omega)|^2\, d\omega \quad . \tag{48}$$

Now consider a specific filter configuration. Let

$$F(s) = s + a \quad .$$

From Eqs. (41) and (42), we have for large t

$$S_x(\omega) = \left(\frac{a\beta}{K} - \sin\theta_o\right) \delta(\omega) + \frac{N_o}{2\pi} \quad , \tag{49}$$

$$\sigma^2_{\hat{E}_o} = \frac{N_o}{2\pi} \int_{-\infty}^{\infty} |L(j\omega)|^2 \, d\omega = N_o \Lambda(O) \quad . \tag{50}$$

Therefore,

$$\lim_{t \to \infty} S_{\hat{E}(t)}(\omega) = \left[\left(\frac{\alpha\beta}{K} - \sin \theta_o \right) \delta(\omega) + \frac{N_o}{2\pi} + \frac{N_o \Lambda(0) \cdot \frac{N_o}{2\pi}}{1 - N_o \Lambda(0)} \right] |L(j\omega)|^2 \quad . \tag{51}$$

The mean of E(t) is the height of the impulse at the origin.

$$\lim_{t \to \infty} \left\langle \hat{E}(t) \right\rangle = \left(\frac{\alpha\beta}{K} - \sin \theta_o \right) . \tag{52}$$

If we let $\theta_o = \sin^{-1} \alpha\beta/K$, then $\left\langle \hat{E}(t) \right\rangle = 0$. Then

$$\lim_{t \to \infty} \sigma^2(\hat{E}(t)) = \left\langle \hat{E}^2(t) \right\rangle = \frac{1}{2\pi} \left[N_o + \frac{N_o^2 \Lambda(0)}{1 - N_o \Lambda(0)} \right] \int_{\infty}^{\infty} |L(j\omega)|^2 \, d\omega \quad . \tag{53}$$

Rewriting, we have

$$\lim_{t \to \infty} \sigma^2[\hat{E}(t)] = \left[N_o + \frac{N_o^2 \Lambda(0)}{1 - N_o \Lambda(0)} \right] \Lambda(0) = N_o \Lambda(0) \left[\frac{1}{1 - N_o \Lambda(0)} \right] . \tag{54}$$

For an arbitrary filter, $1/F(s)$, which has only poles in its transfer function, $N_o \Lambda(0) < 1$ is necessary, but not sufficient, to insure that the function $[1 - N_o \Lambda(s)]$ has no roots in the right half plane.

One notices that as $N_o \Lambda(0) \to 0$, the result in Eq. (51) becomes identical to the results one would obtain using a stationary, linear analysis.

The interpretation of our results is exactly the same as in the first-order case. Above a certain threshold,

$$N_o \Lambda(0) > 1 \quad , \tag{55}$$

the variance becomes unbounded for large time.

Now we must relate our threshold values to the original system parameters. If we define

$$N = \frac{N'_o}{2} \int_{-\infty}^{\infty} |L(j\omega)|^2 \, d\omega = 2A^2 N'_o \cdot \Lambda(0) \quad . \tag{56}$$

(Notice that, for $\theta_o = 0$, $L(j\omega) = \theta_2(j\omega)/\theta_1(j\omega)$, so that $\Lambda(0)$ is exactly the loop bandwidth, as commonly defined.) Thus, an overbound to the threshold inequality can be written as

$$\Lambda(0) \frac{N'_o}{A^2} \equiv \left(\frac{N}{S} \right) < 2 \qquad [2^{nd} \text{ or higher order loops}] \quad , \tag{57}$$

since S, the signal power, equals A^2. We recall that this is only a necessary condition. The complete requirement is that

$$1 - (N_o'/A^2)\,\Lambda(s)$$

have no roots in the right half plane.

In the first-order case, the bandwidth is K/2 cps and the threshold for an unstable variance may be written as

$$\frac{KN}{2A^2} < 1 \longrightarrow \left(\frac{N}{S}\right) < 1 \qquad [1^{st} \text{ order loops}] \quad . \tag{58}$$

The difference between the values in Eqs. (57) and (58) is intuitively disturbing. It is discussed in Sec. III of Appendix C.

It is important to emphasize that this inequality is an upper bound to the exact threshold in the nonlinear system. We recall that in the first-order system, as θ_o increased, the limit on KN_o decreased. In the limit at $\beta = K$, any noise made the system asynchronous. For non-zero θ_o in higher-order system, $L(j\omega)$, and consequently $\Lambda(0)$, is a function of θ_o. In our example,

$$L(j\omega) = \frac{K}{S^2 + S + K\cos\theta_o} \quad . \tag{59}$$

Using tabulated integrals,[8] we can evaluate θ_o as a function of $\cos\theta_o$.

$$\int_{-\infty}^{\infty} |L(j\omega)|^2\, d\omega = \frac{\pi K}{a\cos\theta_o}\ \text{rad/sec} = \frac{K}{2a\cos\theta_o}\ \text{cps} \quad . \tag{60}$$

Thus, the variance threshold decreases as θ_o increases from zero. Therefore, the inequalities (57) and (58) form an upper bound to the system threshold. We see that the variance threshold behavior of first and higher-order systems is essentially the same.

One notes that we have excluded filters with zeros in their transfer function. As shown in Sec. II of Appendix C, by a simple modification of the technique an arbitrary rational transfer function

$$H(s) = \frac{\sum_{j=0}^{N} a_j s^j}{\sum_{i=0}^{M} b_i s^i} \quad , \qquad M - N \geq 2 \tag{61}$$

can be examined. The result for $\langle E^2(t)\rangle$ comes out in exactly the same form. The requirement for a stable variance is that $G(s) = [1 - N_o\,\Lambda(s)]$ has no roots in the right half plane. In general, this stability must be determined from the Routh criterion and gives rise to a set of inequalities which must be satisfied. Clearly, $N_o\,\Lambda(0) < 1$ is necessary, but not sufficient to insure a stable variance.

Up to this point, our primary concern has been to obtain a model of phase-locked loop that could explain the threshold effect. However, our analysis led to an expression for the mean and variance of the error that were good approximations for high signal-to-noise ratios. It is worth while to discuss briefly how this expression could be used to perform a system optimization.

V. APPLICATION TO SYSTEM DESIGN AND OPTIMIZATION

Clearly, the signal-to-noise ratio at threshold is directly related to the bandwidth required for tracking the variations in the input phase. In the previous section, we used the model in Fig. 3 which assumed a constant θ_1. We now want to consider variations in input phase. Therefore, the model in Fig. 2 is more useful.

Assume that the variations in phase are slow. In that case, we may consider only $\theta_1 \ll 1$. Consider the case where $\theta_2 \ll 1$. This means that the error is small.

Then we can write

$$E(t) + \frac{p\,F(p)}{K} E(t) - m_1(t)\,E(t) = m_1(t)\,\theta_1(t) + \frac{1}{K}\,\dot{\theta}_1(t) + m_2(t) \quad . \tag{62}$$

The techniques developed in Appendix C are not useful because the input process and the parameter variation are not independent.

However, if we let θ_2 be the variable, we have

$$\theta_2(t) + \frac{p\,F(p)}{K}\,\theta_2(t) + m_1(t)\,\theta_2(t) = \theta_1(t) + m_2(t) \quad . \tag{63}$$

This differential equation is identical in form to Eq. (42).

From Eq. (47), we have

$$\lim_{t\to\infty} S_{\theta_2}(\omega) = \left[S_x(\omega) + \frac{\sigma^2_{\theta_{20}}(\infty)\,\frac{N_o}{2\pi}}{1 - N_o\,\Lambda(0)} \right] |K(j\omega)|^2 \quad , \tag{64}$$

where

$$K(j\omega) = \frac{1}{\frac{(j\omega)\,F(j\omega)}{K} + 1} \equiv \frac{\theta_2(j\omega)}{\theta_1(j\omega)} \quad , \tag{65}$$

and

$$x(t) = \theta_1(t) + m_2(t) \quad . \tag{66}$$

Then we may write

$$\lim_{t\to\infty} \left\langle \theta_2^2(t) \right\rangle = \int_{-\infty}^{\infty} S_{\theta_2}(\omega)\,d\omega \quad . \tag{67}$$

However, we still desire to minimize $\left\langle E^2(t) \right\rangle$:

$$\left\langle E^2(t) \right\rangle = \left\langle [\theta_1 - \theta_2]^2 \right\rangle = \left\langle \theta_1(t)^2 \right\rangle - 2\left\langle \theta_1(t)\,\theta_2(t) \right\rangle + \left\langle \theta_2(t)^2 \right\rangle \quad . \tag{68}$$

For the case where $\theta_1(\tau)$ and $n_1(\tau)$ are stationary processes, E is not a function of time. Since $\left\langle \theta_1(t)^2 \right\rangle$ is fixed, we wish to minimize

$$F = \left\langle \theta_2(t)^2 \right\rangle - 2\left\langle \theta_1(t)\,\theta_2(t) \right\rangle \quad . \tag{69}$$

We must then evaluate the second term. In Appendix D, 1 it is shown that if $\theta_1(t)$ and $n_1(t)$

are sample functions from stationary, zero mean independent Gaussian processes, the value of this term is given by the following expression:

$$\left\langle \theta_1(t)\ \theta_2(t) \right\rangle \int_{-\infty}^{\infty} \mathrm{Re}\ K(j\omega)\ \Phi_{\theta_1}(j\omega)\, d\omega \tag{70}$$

$$F = \int_{-\infty}^{\infty} \left\{ \left[S_{\theta_1}(\omega) + \frac{N_o}{2\pi} \right] \cdot |K(j\omega)|^2 \right\} d\omega \left[1 + \frac{N_o\,\Lambda(0)}{1 - N_o\,\Lambda(0)} \right] - 2 \int_{-\infty}^{\infty} \mathrm{Re}\left[K(j\omega)\right] S_{\theta_1}(\omega)\, d\omega\ . \tag{71}$$

The problem is to vary the transfer function $K(j\omega)$ such that F is minimized while constraining $(1 - N_o\ \Lambda(s))$ to be stable. One notices that since $\Lambda(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |K(j\omega)|\, d\omega$, the minimization is not a conventional Wiener-Lee filter problem.

There are two ways to approach this minimization. One could attempt to modify the Wiener-Hopf technique to find the optimum linear filter. In this case, it seems easier to specify the form of the filter and vary the parameters to find the minimum F. This approach is outlined briefly in Appendix D. The actual computation of values is not carried out.

VI. CONCLUSIONS

A useful model for a phase-locked loop has been developed. By recognizing the loop as a nonlinear system with randomly time-varying parameters, a logical explanation of the thresholding effect is formulated. Bounds are obtained on the noise level necessary to cause asynchronous behavior.

If the statistical properties of the input are known, the model may be used as a basis for system optimization.

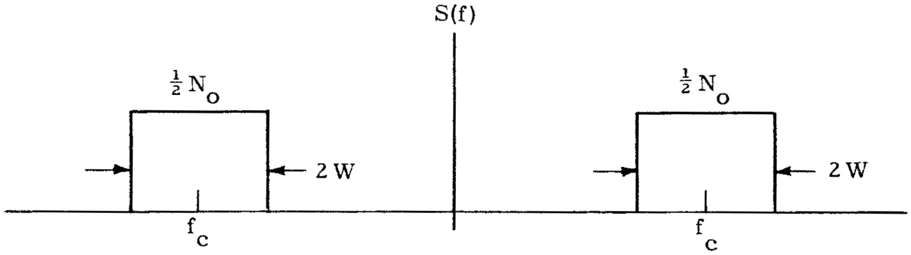
ACKNOWLEDGMENT

Frequent discussions with Dr. Barney Reiffen are gratefully acknowledged. In many cases, his specific suggestions or detailed questions kept the analysis moving.

APPENDIX A
INPUT NOISE CHARACTERISTICS

The input noise is assumed to be a narrow-band Gaussian process. Its characteristics are well known and are summarized below.[5]

The spectrum is:



We may write

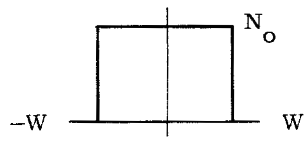
$$n'(t) = n_1'(t) \sin \omega_c t + n_2'(t) \cos \omega_c t \quad ,$$

where $n_1'(t)$ and $n_2'(t)$ are low-frequency random processes, and

$$\overline{n_1'(t)^2} = \overline{n_2'(t)^2} = 2N_o'W \quad ,$$

where N_o' is the mean noise/unit bandwidth in cps.

Then, $n_1'(t)$ and $n_2'(t)$ have spectrums



One notes that the spectrum we have specified is symmetrical about f_c. This insures that $n_1'(t)$ and $n_2'(t)$ are sample functions from independent random processes.

APPENDIX B
COMPUTATION OF EXPECTED VALUES

It is desired to find $\left\langle \exp\left[-K' \int_x^t n_1'(\tau)\, d\tau\right]\right\rangle$, where $n_1'(\tau)$ is a sample function from a white Gaussian process with $\langle n_1'(\tau)\rangle = 0$ and $\phi_{n_1 n_1}(\tau) = N_o\, \delta(\tau)$.

Defining the random variable N_{tx},

$$N_{tx} = \int_x^t n_1'(\tau)\, d\tau \quad . \tag{B-1}$$

One knows (Ref. 5, p. 80) that N_{tx} is a Gaussian random variable and

$$\langle N_{tx} \rangle = 0 \quad , \tag{B-2}$$

and

$$\sigma^2(N_{tx}) = 2[t-x] \int_x^t \left(1 - \frac{\tau}{t-x}\right) [\sigma_{nn}(\tau)]\, d\tau \tag{B-3}$$

or

$$\sigma^2(N_{tx}) = N_o(t-x) \quad . \tag{B-4}$$

One can write the desired average as

$$\left\langle e^{-KN_{tx}} \right\rangle = \int_{-\infty}^{\infty} e^{-KN_{tx}} \cdot \frac{1}{2\pi\, \sigma(N_{tx})}\, e^{N_{tx}^2/-2\sigma^2}\, dN_{tx} \quad .$$

Completing the square, one has

$$\left\langle e^{-KN_{tx}} \right\rangle = e^{\sigma^2 K^2/2} = e^{K^2 N_o (t-x)/2} \tag{B-5}$$

as the desired expectation.

APPENDIX C
DERIVATION OF PROPERTIES OF HIGHER-ORDER SYSTEMS

I. BASIC DERIVATION*

We are concerned with equations of the form:

$$L(y) = \frac{d^n y}{dt^n} + \frac{d^{n-1}(a_{n-1}y)}{dt^{n-1}} + \ldots a_o(t)\, y = r(t) \quad . \tag{C-1}$$

The $a_o, \ldots, a_{n-1}$ may be sample functions from strict sense stationary processes or constants (a_k must possess k derivatives). A Taylor series expansion of the solution is not applicable because, in general, the solution is not analytic. However, an expansion due to Carson[9] is valid.

First, divide $a_k(t)$ into a constant and random term

$$a_k(t) = b_k(t) + C_k, \quad \left\langle b_u(t) \right\rangle = 0 \quad . \tag{C-2}$$

Define

$$M(y) = y^{(n)} + C_{n-1}y^{(n-1)} + \ldots C_o y \quad , \tag{C-3}$$

$$N(y) = \frac{d^{n-1}}{dt^{n-1}}\,(b_{n-1}y) + \ldots + b_o(t)\, y \quad . \tag{C-4}$$

Thus,

$$L(y) = M(y) + N(y) \quad . \tag{C-5}$$

Let $k(t-\tau)$ be the kernel corresponding to M. Thus,

$$\int_0^t k(t-\tau)\, M\left[y(\tau)\right] d\tau = y(t) \quad . \tag{C-6}$$

Applying to Eq. (C-5), we have

$$\int_0^t k(t-\tau)\, M\left[y(\tau)\right] d\tau + \int_0^t k(t-\tau)\, N\left[y(\tau)\right] d\tau = \int_0^t k(t-\tau)\, r(\tau)\, d\tau \quad . \tag{C-7}$$

Defining $y_o(t) = \int_0^t k(t-\tau)\, r(\tau)\, d\tau$, we have

$$y(t) = y_o(t) - \int_0^t k(t-\tau)\, N\left[y(\tau)\right] d\tau \quad . \tag{C-8}$$

For zero initial conditions and the order of M greater than N, we can write

---

*This derivation is due to A. Rosenbloom. Our treatment follows his results in Ref. 6.

$$\int_0^t k(t-\tau)\, N[y(\tau)]\, d\tau = \int_0^t N^*[K(t-\tau)]\, y(\tau)\, d\tau \quad , \tag{C-9}$$

where N^* is adjoint differential operator of N.

$$N^* = \sum_{j=0}^{n} (-1)^j\, b_j(\tau)\, \frac{d^j}{d\tau^j} \quad . \tag{C-10}$$

[The validity of (C-9) is shown in Ref. 11, p. 211.]

Rosenbloom states that the only case that is easily solvable is that where all $b_j(t)$ but one are zero, and the non-zero $b_j(t)$ is a sample function from a white, Gaussian process. The obvious extension (which Rosenbloom was undoubtedly aware of) is the case where all $b_j(t)$ are related to each other by constant multipliers. This case will be demonstrated in Sec. II of Appendix C.

In this case, Eq. (C-8) becomes

$$y(t) = y_o(t) - \int_0^t k(t-\tau)\, b_o(\tau)\, y(\tau)\, d\tau \quad . \tag{C-11}$$

If $b_o(\tau)$ and $y(\tau)$ were uncorrelated, one could write

$$\langle y(t) \rangle = y_o(t) - \int_0^t k(t-\tau) \langle b_o(\tau) \rangle \langle y(\tau) \rangle\, d\tau \quad . \tag{C-12}$$

Then, $\langle y(t) \rangle = y_o(t)$, since $\langle b_o(\tau) \rangle = 0$ and, if $b_o(\tau_1)\, b_o(\tau_2)$ and $y(\tau_1)\, y(\tau_2)$ were uncorrelated, then

$$\langle y^2 \rangle = y_o^2 + \int_0^t \int_0^t k(t-\tau_1)\, k(t-\tau_2)\, \phi_b(\tau_1-\tau_2) > y(\tau_1)\, y(\tau_2) > d\tau_1\, d\tau_2 \quad . \tag{C-13}$$

These assumptions are certainly not _a priori_ obvious. However, for $k(0) = 0$, Rosenbloom[10] has proved that they are true.

Heuristically, one can see this by considering the series solution to (C-11):

$$y(t) = y_o(t) + \sum_{i=1}^{\infty} y_i(t) \quad , \tag{C-14}$$

where

$$y_i(t) = \int_0^t k(t-\tau)\, b_o(\tau)\, y_{i-1}(\tau)\, d\tau \quad . \tag{C-15}$$

We are concerned with $\langle b_o(t)\, y_i(t) \rangle$.

For i = 1,

$$\left\langle b_o(t)\,y_1(t)\right\rangle = \int_0^t k(t-\tau)\left\langle b_o(\tau)\,b_o(t)\right\rangle y_o(\tau)\,d\tau \quad . \tag{C-16}$$

When $b_o(t)$ is the sample function from a white Gaussian noise process, we have

$$\left\langle b_o(t)\,y_1(t)\right\rangle = \int_0^t k(t-\tau)\,\delta(t-\tau)\,y_o(\tau)\,d\tau = 0 \quad \text{for} \quad k(0) = 0 \quad . \tag{C-17}$$

For $i = 2$,

$$\left\langle b_o(t)\,y_2(t)\right\rangle = \int_0^t k(t-\tau)\left\langle b_o(t)\,b_o(\tau)\,y_1(\tau)\right\rangle d\tau \quad , \tag{C-18}$$

but $\left\langle b_o(t)\,b_o(\tau)\;y_1(\tau)\right\rangle$ factors for a Gaussian process and the resultant expectation is zero for $k(0) = 0$. Having established Eqs. (C-12) and (C-13), the rest of the work is straightforward.

Since $b_o(t)$ is white noise, $\phi_b(\tau_1 - \tau_2) = N_o\,\delta(\tau_1 - \tau_2)$ and (C-13) becomes

$$\left\langle y^2\right\rangle = y_o^2 + N_o\;k^2(t)\;\oplus\left\langle y^2\right\rangle \;[\oplus \text{ denotes convolution}] . \tag{C-19}$$

Defining the transforms

$$\mathcal{L}\,[y_o^2(t)] = \Gamma(s) \quad , \tag{C-20}$$

$$\mathcal{L}\,[k^2(t)] = \Lambda(s) \quad , \tag{C-21}$$

we have

$$\left\langle y^2(t)\right\rangle = \mathcal{L}^{-1}\left[\frac{\Gamma(s)}{1 - N_o\,\Lambda(s)}\right] \quad . \tag{C-22}$$

Thus, stability of the variance requires that $1 - N_o\,\Lambda(s)$ has no roots in the right half plane. First, consider the case for a deterministic $r(t)$. Here, we define

$$\lim_{t\to\infty}\; y_o^2(t) = M^2 \quad . \tag{C-23}$$

Then,

$$\lim_{t\to\infty}\;\left\langle y^2(t)\right\rangle = \frac{M^2}{1 - N_o\,\Lambda(0)} \quad , \tag{C-24}$$

and

$$\lim_{t\to\infty}\;\sigma_y^2(t) = \frac{N_o\,\Lambda(0)\,M^2}{1 - N_o\,\Lambda(0)} \quad . \tag{C-25}$$

In a similar fashion, one can show that if $r(t)$ is a random process, then

$$\left\langle y(t)\right\rangle = \left\langle y_o(t)\right\rangle \quad , \tag{C-26}$$

and

$$\left\langle y^2(t) \right\rangle = \frac{\left\langle y_o^2(t) \right\rangle}{1 - N_o \Lambda(0)} \quad . \tag{C-27}$$

If r(t) contains both a deterministic and random component, one can find an expression for the spectrum which is valid for large t. The result is

$$S_y(\omega) = \left\{ S_r(\omega) + \frac{\sigma_{y_o}^2(\infty)\frac{N_o}{2}}{1 - N_o \Lambda(0)} \right\} |k(j\omega)|^2 \quad . \tag{C-28}$$

II. FILTERS WITH ZEROS

Now consider the case where the transfer function of the filter in Fig. 5 is a ratio of two polynomials:

$$H(s) = \frac{A(s)}{B(s)} \quad , \tag{C-29}$$

where the order of A(s) is at least two less than the order of B(s). Then, Eq. (C-1) becomes

$$p\,B(p)\left\{\theta_1 - E(t)\right\} = A(p)\,E(t) = \left[1 + p_1(t)\right] A(p)\,p_2(t) \quad , \tag{C-30}$$

where we have set $\theta_o = 0$, for convenience. Rearranging, we have

$$\left[p\,B(p) + A(p)\right] E(t) + A(p)\left[p_1(t)\,E(t)\right] = p\,B(p)\,\theta_p - A(p)\,p_2(t) \quad . \tag{C-31}$$

Now we define

$$r(t) \equiv p\,B(p)\,\theta_1(t) + A(p)\,p_2(t) \quad . \tag{C-32}$$

Let k(t) be the kernel corresponding to the operator $M \equiv p\,B(p) + A(p)$. If we write

$$A(p) = a_o + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \ldots \quad , \tag{C-33}$$

we see that from (C-10),

$$N^* = \sum_{j=0}^{n} (-1)^j \cdot p_1(\tau)\,a_j \frac{d_j}{d\tau^j} \quad . \tag{C-34}$$

Thus, the term in (C-9) becomes

$$\int_0^t \left\{ \sum_{j=0}^{n} (-1)^j a_j \frac{d^j}{d\tau^j}\left[k(t-\tau)\right] \right\} p_1(\tau)\,y(\tau)\,d\tau \quad , \tag{C-35}$$

but the kernel defined by the expression inside the braces is simply

$$k'(t) = \mathcal{L}^{-1} \frac{A(s)}{sB(s) + A(s)} \quad . \tag{C-36}$$

Thus (C-11) becomes

$$y(t) = y_o(t) - \int_0^t k'(t-\tau)\, p_1(\tau)\, y(\tau)\, d\tau \quad , \tag{C-37}$$

where

$$y_o(t) = \mathcal{L}^{-1}\left[\frac{sB(s)}{sB(s) + A(s)}\,\theta_1(s) - \frac{A(s)}{sB(s) + A(s)}\,P_2(s)\right] \quad . \tag{C-38}$$

One notices that $y_o(t)$ is the normal output expression for a linear system with two inputs and that $k'(t)$ is the kernel corresponding to the transfer function $\theta_2(s)/\theta_1(s)$.

III. COMMENTS ON THE SOLUTION

In the first two sections, we have shown how to analyze the linearized model for all filters such that $k'(0) = 0$. This is the same as the restriction that if

$$k(s) = \frac{\sum_{j=0}^{N} a_j s^j}{\sum_{i=0}^{M} b_i s^i} \quad , \tag{C-39}$$

then $M - N \geq 2$.

Intuitively, this seems to be an unnecessary restriction. Consider the case

$$K_1(s) = \frac{S + B}{S^2} \quad . \tag{C-40}$$

Here $k_1(0) = \lim_{s \to \infty} \frac{S + B}{S^2} \cdot S = 1.$ Now approximate $K_1(s)$ by

$$K_2(s) = \frac{S + B}{S^2\left(\frac{S}{\alpha_n} + 1\right)} \quad . \tag{C-41}$$

Here

$$k_2(0) = 0 \quad . \tag{C-42}$$

Therefore, the developments in Secs. I and II of Appendix C are valid for system 2, but not for system 1.

As α_n approaches ∞, the impulse response $k_{2_n}(t)$ approximates $k_1(t)$ in some sense. For each impulse response $k_{2_n}(t)$, an output $y_{2_n}(t)$ is obtained.

One must then show how well $y_{2_n}(t)$ can be made to approximate $y_1(t)$ by choosing α_n large enough. For the case where $y_2(t)$ has a finite mean and variance, one can show that $y_{2_n}(t)$ converges to $y_1(t)$ in probability.

However, one can show examples where the outputs of the two systems do not converge in the mean. For suitable N_o, the first-order system K/S has a divergent mean. The approximating system $K/[s(\frac{s}{\alpha_n} + 1)]$ has a well-defined mean. Therefore, from the standpoint of our

model, a first-order system is not a proper sub-class of a higher-order system. It appears that the difficulty arises because the white noise solution uses $k(0) = 0$ as a fundamental assumption. In a physical system, since noise cannot be perfectly white, one would expect the two outputs to be the same.

APPENDIX D
DETAILS OF SYSTEM OPTIMIZATION

I. EVALUATION OF CROSS-CORRELATION TERM

We are concerned with the evaluation of the term $\langle \theta_1(t)\,\theta_2(t) \rangle$ for large t. In particular, we want to consider the case where $\theta_1(t)$, $m_1(t)$ and $m_2(t)$ are sample functions from stationary, independent Gaussian processes with zero means.

From Appendix C, we know that we can rewrite Eq. (60) as the solution of an integral equation,

$$\theta_2(t) = \int_0^t k(t-\tau)\left[\theta_1(\tau) + m_2(\tau)\right] d\tau + \int_0^t k(t-\tau)\, m_1(\tau)\, \theta_2(\tau)\, d\tau \quad , \tag{D-1}$$

where

$$k(t) = \mathcal{L}^{-1} \frac{K}{SF(s) + K} \quad . \tag{D-2}$$

One can write

$$\lim_{t\to\infty} \langle \theta_1(t)\,\theta_2(t) \rangle = \lim_{t\to\infty} \langle \theta_1(t)\,\theta_2(t+\mu) \rangle \Big|_{\mu=0} = \phi_{\theta_1\theta_2}(\infty, 0) \tag{D-3}$$

We want to prove that the second term on the right side of Eq. (D-1) does not contribute anything to the correlation function.

Considering just this term,

$$\phi^{(2)}_{\theta_1\theta_2}(\infty, 0) = \lim_{t\to\infty} < \theta_1(t) \int_{-\infty}^t k(t-\tau)\, m_1(\tau)\, \theta_2(\tau)\, d\tau > , \tag{D-4}$$

or equivalently,

$$\phi^{(2)}_{\theta_1\theta_2}(\infty, 0) = \int_{-\infty}^t k(t-\tau)\, \langle \theta_1(t)\, m_1(\tau)\, \theta_2(\tau) \rangle\, d\tau \quad . \tag{D-5}$$

If $\theta_1(t)$, $m_1(\tau)$, $\theta_2(\tau)$ were sample functions from jointly Gaussian random processes, we could write

$$\langle \theta_1(t) m_1(\tau)\theta_2(\tau)\rangle = \langle \theta_1(t)\rangle \langle m_1(\tau)\theta_2(\tau)\rangle + \langle m_1(\tau)\rangle\langle \theta_1(t)\theta_2(\tau)\rangle + \langle \theta_2(\tau)\rangle \langle \theta_1(t) m_1(\tau)\rangle \; . \tag{D-6}$$

Then, since

$$\langle \theta_1(t)\rangle = \langle m_1(\tau)\rangle = 0 \quad , \tag{D-7}$$

and

$$\langle \theta_1(t)\, m_1(\tau) \rangle = \langle \theta_1(t)\rangle \langle m_1(\tau)\rangle = 0 \quad , \tag{D-8}$$

therefore,

$$\phi^{(2)}_{\theta_1\theta_2}(\infty, 0) = 0 \quad . \tag{D-9}$$

It is only necessary to show that $\theta_1(t)$, $m_1(\tau)$, $\theta_2(\tau)$ are jointly Gaussian. We can write

$$\theta_2(t) = \theta_o(t) + \sum_{i=1}^{\infty} \theta_i(t) \quad , \tag{D-10}$$

where

$$\theta_o(t) = \int_{-\infty}^{t} k(t-\tau)\left[\theta_1(\tau) + m_2(\tau)\right] d\tau \quad , \tag{D-11}$$

and

$$\theta^{(i)}(t) = \int_{-\infty}^{t} k(t-\tau)\, m_1(\tau)\, \theta^{(i-1)}(\tau)\, d\tau \quad . \tag{D-12}$$

From Ref. 5, we know that:

(1) $\theta_o(t)$ and $\theta_1(\tau)$ are jointly Gaussian.

(2) Considering $k(t-\tau)\, m_1(\tau)$ as the time-varying kernel, $\theta^{(i)}(t)$ and $\theta^{(i-1)}$ are jointly Gaussian.

Therefore, $\theta_2(t)$ and $\theta_1(t)$ are jointly Gaussian.

(3) Considering $k(t-\tau)\, \theta^{(i-1)}(\tau)$ as the time-varying kernel, $\theta_2^{(i)}(t)$ and $m_1(t)$ are jointly Gaussian.

Thus, $\theta_2(t)$ and $m_1(t)$ are jointly Gaussian.

Since $m_1(t)$ and $\theta_1(t)$ are independent, it follows that

$$p\left[\theta_1(t_1)\, \theta_2(t_2)\, m_1(t_2)\right] = p\left[\theta_1(t_1) \,|\, \theta_2(t_2)\right]\, p\left[\theta_2(t_2)\, m_1(t_3)\right] \quad , \tag{D-13}$$

and the three processes are jointly Gaussian.

Evaluating the first term, we have

$$\phi_{\theta_1\theta_2}(\infty, \mu) = \lim_{t\to\infty} \theta_1(t) \int_{-\infty}^{t} k(\tau)\left[\theta_1(t-\tau) + m_2(t-\tau)\right] d\tau \tag{D-14}$$

$$= \int_{-\infty}^{t} k(\tau) \cdot \lim_{t\to\infty} \left\langle \theta_1(t)\left[\theta_1(t+\mu-\tau) + m_2(t+\mu-\tau)\right] \right\rangle \quad ,$$

$$\phi_{\theta_1\theta_2}(\infty, \mu) = \int_{-\infty}^{\infty} k(\tau)\left[\phi_{\theta_1\theta_1}(\mu-\tau)\right] d\tau \quad , \tag{D-15}$$

or

$$\Phi_{\theta_1\theta_2}(j\omega) = K(j\omega)\, \Phi_{\theta_1\theta_1}(j\omega) \quad , \tag{D-16}$$

and the desired quantity is

$$\langle \theta_1(t)\,\theta_2(t) \rangle = \int_{-\infty}^{\infty} K(j\omega)\,\Phi_{\theta_1\theta_1}(j\omega)\,d\omega = \int_{-\infty}^{\infty} \mathrm{Re}\left[K(j\omega)\right]\,\Phi_{\theta_1\theta_1}(j\omega)\,d\omega \quad , \tag{D-17}$$

since the imaginary part of the transform of a realizable system is odd.

II. OPTIMIZATION PROCEDURE

The technique of optimizing a fixed-form system with variable parameters is straightforward but laborious.

The basic idea[8] is to express the error as a definite integral of the form

$$E = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{c(s)\,c(-s)}{d(s)\,d(-s)}\,ds \quad , \tag{D-18}$$

where c(s) and d(s) are polynomials which contain the variable parameters. Thus,

$$c(s) = \sum_{i=0}^{n} c_i s^i \quad , \tag{D-19}$$

$$d(s) = \sum_{i=0}^{n} d_i s^i \quad . \tag{D-20}$$

From tabulated integrals, one can then write E as a function of c_i and d_i.

As an example, let the filter transfer function be

$$H(s) = \frac{s+\beta}{s(s+\alpha)} \quad . \tag{D-21}$$

Then

$$K(s) = \frac{K(s+\beta)}{s^3 + \alpha s^2 + Ks + \beta} \quad , \tag{D-22}$$

and

$$\mathrm{Re}\left[K(s)\right] = \frac{-Ks^4 + (K\beta\alpha - K^2)\,s^2 + K\beta^2}{|s^3 + \alpha s^2 + Ks + \beta|^2} \quad . \tag{D-23}$$

With some manipulation and change of variable, this can be put into the form

$$\mathrm{Re}\left[K(s)\right] = \left|\frac{c_2 s^2 + c_o}{d_3 s^3 + d_2 s^2 + d_1 s + d_o}\right|^2 \quad . \tag{D-24}$$

Let the phase spectrum be

$$\Phi_{\theta_1}(s) = \frac{A}{-s^2 + B^2} \quad . \tag{D-25}$$

Substituting into Eq. (71) of the text and using tabulated integrals, we have

$$F = f(K, \beta, \alpha; \Phi_{\theta_1}, A, B) \quad . \tag{D-26}$$

Now E must be minimized subject to the constraint that

$$g(K, \beta) = N_o \Lambda(0) = N_o \int_{-\infty}^{\infty} |K(j\omega)|^2 \, d\omega < 1 \quad . \tag{D-27}$$

In our particular case,

$$g(K, B) = N_o \left[\frac{K(K+\beta)}{\alpha + K} \right] \quad . \tag{D-28}$$

An analytic minimization using Lagrange multipliers is too difficult.

The easiest approach appears to be to specify A, B and α and plot E as a function of K and β. Then the minimum subject to the constraint of Eq. (D-27) can be found graphically.

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